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Phase transitions in lattices of coupled chaotic maps and their dependence on the local Lyapunov exponent

Francisco Sastre* and Gabriel Pérez†
Departamento de Física Aplicada, Centro de Investigación y de Estudios Avanzados del Instituto Politécnico Nacional, Unidad Mérida, Apartado Postal 73 ''Cordemex,'' 97310 Mérida, Yucatán, México
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We study the continuous phase transitions in lattices of chaotic maps recently found by Miller and Huse. It is believed that in these lattices the order-disorder transition is generated by competition between (ordering) diffusion and (disordering) local chaos. As a test of this idea, we check whether the local Lyapunov exponent of the system behaves as a univocal extra control parameter for criticality. We verify the presence of phase transitions for a whole family of maps, both as function of coupling and of an internal parameter related to their chaoticity. We find that the critical coupling parameter is not a one-to-one function of the local Lyapunov exponent, which implies that this exponent cannot be used in general as control parameter for the transition. [S1063-651X(98)08904-1]

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I. INTRODUCTION

There has been a growing interest in the dynamics of locally coupled chaotic systems, since the discovery of long-range collective behavior in high-dimensional cellular automata [1,2] and in coupled map lattices [3]. A remarkable example of this type of coherent evolution was found by Miller and Huse [4], who considered a two-dimensional (2D) lattice of odd-symmetric chaotic maps, and found a continuous phase transition, which originally seemed to fit within the 2D Ising universality class. This contradicted previous theoretical works [5], that had shown that in general collective nontrivial phenomena could not happen in locally coupled chaotic systems, with the exception of period-2 oscillations. In particular, for the case of phase transitions, it was argued that droplets of any given phase could not grow beyond certain size, since their growth (i.e., the ordering process) was at most linear in time, while local (disordering) fluctuations, which were chaotic in origin, went exponentially fast in time.

In this system a diffusive dynamics is introduced by a coupling between nearest neighbors. This interaction is linear, and is quantified by a real coupling parameter. In the map, due to its odd symmetry, the typical orbit has an equal probability of falling in positive or negative values, so when we take the sign of each site as a local order parameter (as in the Ising model) its time average is zero for isolated maps. The time average of the order parameter (defined as the sum of the local order parameter) is zero in the absence of coupling, and remains zero for low coupling values. Above a critical value of the coupling, the order parameter becomes nonzero. These particular phase transitions, and some others that are closely related, were explored by Marcq, Chaté, and Manneville, who found that they do not fall entirely in the 2D Ising universality class [6]. In particular, the exponent ν associated with the divergence of the correlation length was found to be $\nu=0.887(18)$, where the number(s) between parentheses corresponds to the uncertainty in the last digit(s) of the quantity. This value is somewhat smaller than that of the Ising model, $\nu_{\rm Ising}=1$. This difference in universality class seem to be related to the simultaneous updating used in the realization of the system, a type of updating that is not possible for Monte Carlo simulations of equilibrium statistical models [7].

Here we are concerned with the relationship between local chaos and the location of the critical coupling. Assume that we start the evolution of our lattice with an ordered configuration. For identical maps, the source of disorder is their chaoticity, which introduces fluctuations in the order parameter at small scales. If the coupling is not large enough, these fluctuations grow and destroy the large scale order of the system. Therefore, one may expect that lattices made of very chaotic oscillators will have large critical couplings, while a small coupling should be enough to order lattices made of almost nonchaotic elements. In order to test this idea, one has to decide first how to give a quantitative meaning to the expressions ''very chaotic'' and ''almost nonchaotic'' we just used. An approach to this point was given in terms of the Lyapunov dimension of the lattice [8], but the results were not conclusive. In this work we will consider instead a simpler point of view: Taking as the source of fluctuations the chaoticity of the maps, one may as well take as a measure of chaos their own Lyapunov exponents. In favor of this approach we have the fact that it is difficult—at least for our present understanding of the phenomena—to decide whether changes in the global measures of chaos are to be taken as control parameters, like the usual quantities (volume, temperature) that one consider as experimentally controllable in ordinary statistical mechanics, or as response functions (like specific heat, compressibility, etc.). For the local Lyapunov exponent this difficulty does not exist.

II. MODEL

The model proposed by Miller and Huse consists of a two-dimensional lattice of diffusively coupled maps, where

*Electronic address: sastre@kin.cieamer.conacyt.mx
†Electronic address: gperez@kin.cieamer.conacyt.mx

the maps are piecewise linear and are defined by

$$\phi(y)=\begin{cases} -2-3y & \text{for } -1\leqslant y\leqslant -\dfrac{1}{3} \\ 3y & \text{for } -\dfrac{1}{3}<y<\dfrac{1}{3} \\ 2-3y & \text{for } \dfrac{1}{3}\leqslant y\leqslant 1. \end{cases} \tag{2.1}$$

This map is odd symmetric, and its Lyapunov exponent is equal to ln 3. A typical trajectory in this map spends, in average, equal times in the $y<0$ and the $y>0$ ranges. The 2D coupled system obeys the discrete time evolution rule given by

$$y_{i,j}^{t+1}=(1-4g)\phi(y_{i,j}^{t})+g[\phi(y_{i,j+1}^{t})+\phi(y_{i,j-1}^{t}) + \phi(y_{i+1,j}^{t})+\phi(y_{i-1,j}^{t})], \tag{2.2}$$

where t indicates temporal evolution, the indexes i and j are for the site on the lattice, and g is the diffusive coupling parameter. Since the values of $y_{i,j}$ need to be constrained to the interval $[-1,1]$, we require that $0\leqslant g\leqslant \frac{1}{4}$. When $g=0$ each site is isolated, and the global Lyapunov exponent of the system corresponds to the Lyapunov exponent of the map.

In order to make an analogy with the equilibrium Ising models, an order parameter is defined in the following way: An instantaneous order parameter m_L^t is given by

$$m_L^t=\frac{1}{N}\sum_{i,j} \operatorname{sgn}(y_{i,j}^t). \tag{2.3}$$

The sum is over all lattice sites, N is the number of sites, and the subscript L corresponds to the lateral lattice size ($N=L^2$). From here one obtains the order parameter as a time average,

$$\langle m_L\rangle=\frac{1}{t_n+1}\sum_{t=t_0}^{t=t_0+t_n} |m_L^t|, \tag{2.4}$$

where t_0 is a transient and t_n is the time interval over which the average is taken. Finally, a susceptibility χ_L is given by the definition

$$\chi_L=L^2\left(\frac{1}{t_n+1}\sum_{t=t_0}^{t=t_0+t_n}(|m_L^t|-\langle m_L\rangle)^2\right)^{1/2}, \tag{2.5}$$

following the definition of generalized susceptibilities of thermodynamics quantities of the standard statistical mechanics [9].

For zero coupling, the chaotic character of the maps ensures that the order parameter is zero. As the value of g is increased, a continuous phase transition is found, with a critical value $g_c=0.205\,34(2)$ [6,10,11]. The critical exponents of this system are not quite those corresponding to the 2D Ising model, giving in this way a universality class for simultaneous updating [6]. In case that the actual values of the variables $y_{i,j}$ be used instead of their signs, one finds just a change in the size of $\langle m_L\rangle$. This does not affect the critical behavior at all [12].

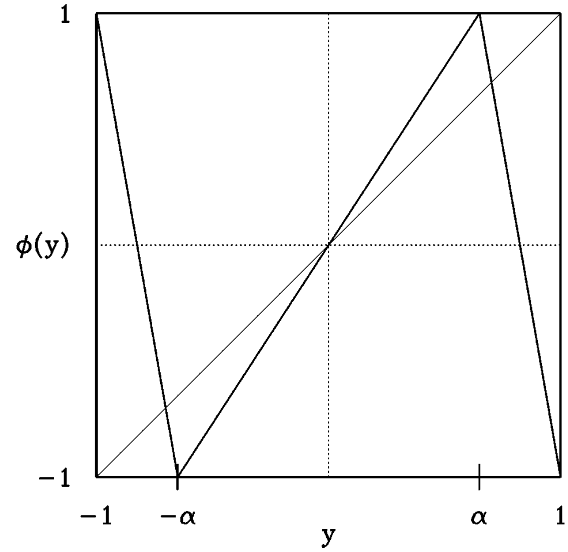

FIG. 1. Local map of the extended Miller-Huse model. We keep the original odd-symmetric form but change the slopes. With this it is possible to change continuously the Lyapunov exponent. Here $\alpha=0.65$.

A. Extended Miller-Huse model

In order to explore the relationship between local chaos and the phase transition, we need to find a family of chaotic maps whose Lyapunov exponent can be changed continuously, which means that it is better not to use maps with quadratic extrema. We also want to keep the odd symmetry of the map. One way of satisfying these conditions is to change the slopes of the Miller-Huse piecewise linear map without changing its odd parity. We propose the map shown in Fig. 1,

$$\phi(y)=\begin{cases} \dfrac{2}{\alpha-1}y+\dfrac{\alpha+1}{\alpha-1} & \text{for } -1\leqslant y\leqslant -\alpha \\ \dfrac{1}{\alpha}y & \text{for } -\alpha<y<\alpha \\ \dfrac{2}{\alpha-1}y-\dfrac{\alpha+1}{\alpha-1} & \text{for } \alpha\leqslant y\leqslant 1, \end{cases} \tag{2.6}$$

where we introduce the internal parameter α bounded to $[0,1]$, and the updating follows Eq. (2.2). When $\alpha=0$, the map degenerates into two lines, both with slope $|\phi'(y)|=2$, and the Lyapunov exponent is therefore $\lambda(\alpha=0)=\ln 2$. In the other extreme of the range the map becomes diagonal, and we obtain $\lambda(\alpha=1)=0$. The Miller-Huse model corresponds to $\alpha=\frac{1}{3}$, with $\lambda(\alpha=\frac{1}{3})=\ln 3$.

B. Lyapunov exponent as function of α

One factor that makes this choice of the map a convenient one is that its Lyapunov exponent can be given as an exact analytical expression in the internal parameter α. This is done using the invariant distribution for the map, which can

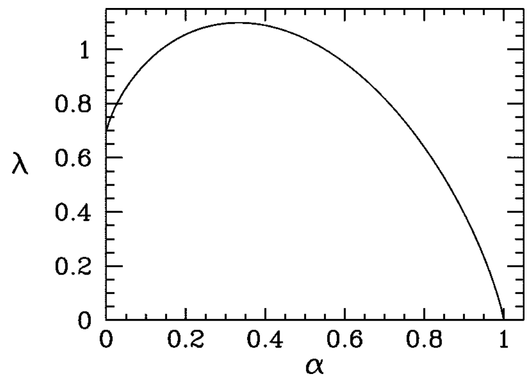

FIG. 2. Local Lyapunov exponent vs internal parameter α. The extremes values of α are as expected. The maximum in λ corresponds to $\alpha=\frac{1}{3}$, which gives the original Miller-Huse map.

be calculated using the method given in Ref. [13] (see also Refs. [14,15]). The result for this invariant distribution is $\rho(y)=\frac{1}{2}$. Integrating the slopes with the invariant distribution, we obtain

$$\lambda=(1-\alpha)\ln\left|\frac{2}{\alpha-1}\right|-\alpha\ \ln\ \alpha. \tag{2.7}$$

The behavior of $\lambda(\alpha)$ is continuous and smooth (Fig. 2),

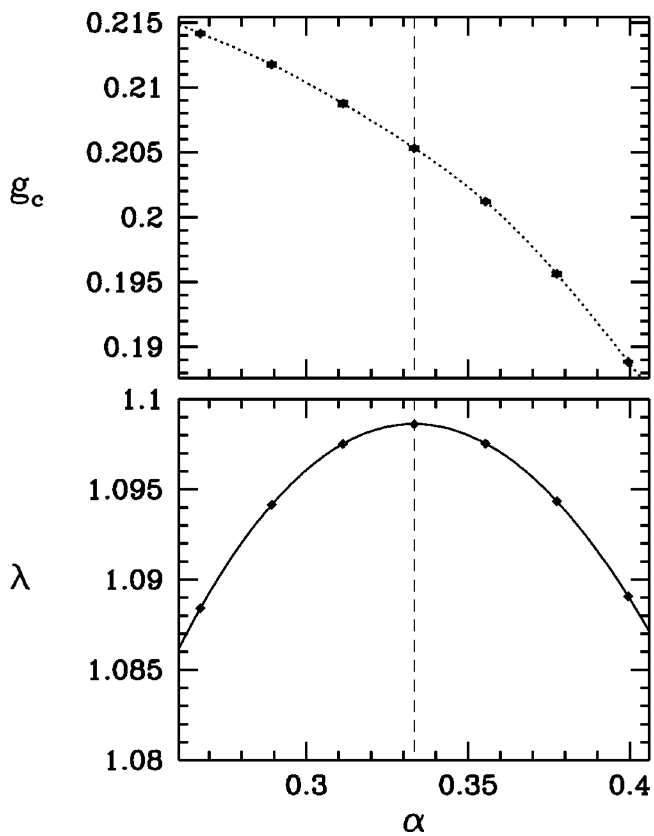

FIG. 3. Critical coupling and local Lyapunov exponent as a function of α. There are two values of the critical coupling for same value of the local Lyapunov exponent. For visualization, we use a spline in the upper graph (dot line). The dashed line marks the value $\alpha=\frac{1}{3}$. Error bars for the critical coupling are smaller that the markers used.

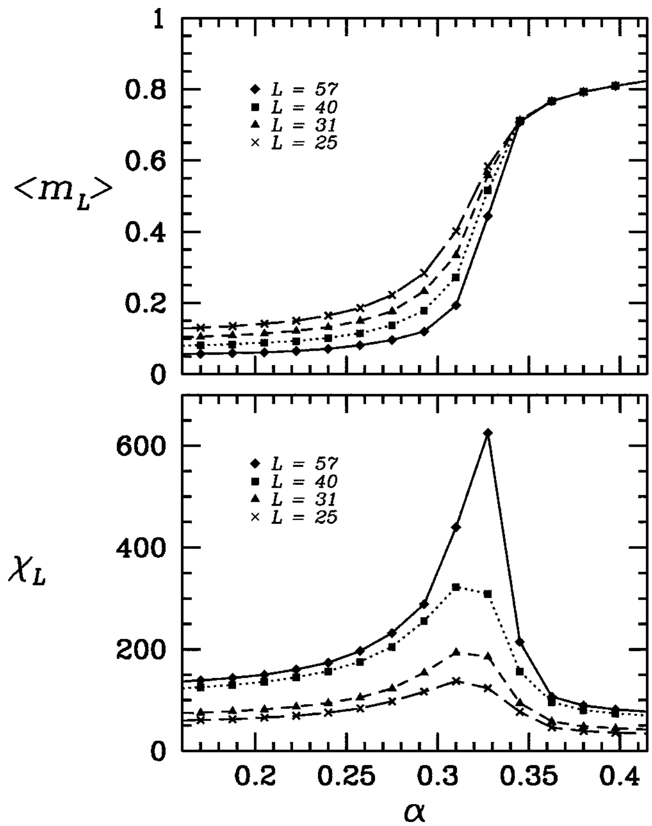

FIG. 4. Order parameter and susceptibility as functions of α. The coupling parameter is fixed at $g=0.205\,34(2)$. A continuous phase transition can be observed, where in both sides of the transition point the local Lyapunov exponent decreases. From cumulants (not shown) we obtain $\alpha_c=0.3333(1)$.

and its values for $\alpha=0$, 1, and $\frac{1}{3}$ are as expected. Notice that the Miller-Huse map has the maximum value possible for λ.

III. PHASE TRANSITIONS

The goal now is to find the relation between the local Lyapunov exponent and the critical coupling (g_c). Numerical simulations were carried out to establish this dependence, and the infinite-size transition point was determined using the cumulant method [16,17], which is based on the fourth order cumulant (U_L) given by

$$U_L=1-\frac{\langle(m_L^t)^4\rangle}{3\langle(m_L^t)^2\rangle^2}. \tag{3.1}$$

As the control parameter tends to the critical point, $U_L\to U^*$, where U^* is a value independent of the size system. The estimated value for two-dimensional Ising models is $U^*\simeq 0.61-0.6116$ [16].

A. Critical coupling versus local Lyapunov exponent

We have calculated the critical coupling for coupled lattices of maps of type (2.6), for seven values of α close to (and including) $\alpha=\frac{1}{3}$. These critical values were located using the already mentioned cumulant method. For this we used lattices of sizes $L=25$, 31, 40, and 57, with helical boundary conditions [18]. The transient and integration times changed for different lattice sizes. In all cases we checked that both times were long enough for convergence. Typical values were above 10^5 and 10^6 iterations for transient and run times, respectively. We assigned a random number, with uniform distribution within the range $[-0.95, 0.95]$, on each lattice site as an initial condition. We located the crossing points of the different cumulant curves, which were approximated using parabolic fitting for the five points closest to the crossing. These crossings clustered on very narrow g ranges. The results are given in Fig. 3, and it is clear from this figure that the critical coupling for the lattice is not a one-to-one function of the local Lyapunov exponent. Therefore, the local chaoticity is not determinant on the long range behavior of the lattice.

B. Behavior as a function of α

As an extra test that the critical coupling for the lattice is not a one-to-one function of the local Lyapunov exponent, we have verified the existence of a phase transition as a function of α for a given value of g. This phase transition is suggested by the phase diagram given in Fig. 3, where one should get a disorder-order transition as α is increased for a given g.

Here we have chosen for g the value given for g_c in Ref. [11] for the Miller-Huse model. According to Fig. 3, a continuous phase transition should be present at $\alpha=\frac{1}{3}$, with the ordered phase appearing for $\alpha>\frac{1}{3}$. Notice that for these values we have that the local maps become less chaotic in both sides of the expected critical α.

The calculations were carried out with the same lattice sizes, transient and runtime of the previous ones. The results are given in Fig. 4, which shows both the order parameter and the susceptibility for this lattice. Their behavior is consistent with a continuous phase transition as α grows. The critical value of α was located using cumulants, and gave $\alpha_c=0.3333(1)$, as expected.

IV. CONCLUSIONS AND FINAL COMMENTS

Starting from a heuristic analogy between chaotic fluctuations in our system and thermal fluctuations in a thermodynamical system in equilibrium, one may have expected the local Lyapunov exponent to behave like a temperature, being in this way a well defined control parameter for coupled chaotic lattices. Contrary to this expectation, we have found that the relationship between local Lyapunov exponents and critical couplings for phase transitions in chaotic lattices is not one to one. Therefore, two systems with the same coupling, and whose elements are equally chaotic, can have completely different collective behaviors. This either means that the use of a local measure for chaos is too naive, or that chaos itself, although a needed element for the order-disorder transition in these systems, is less important for its phase diagram than some other properties of the model. As an extra confirmation of this assertion, we have shown how the same order-disorder phase transition can also happen as a function of internal parameters of the model, for a situation where the local maps become less chaotic at both sides of the critical point.

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